# COMPUTER RECREATIONS

## *A matter fabricator provides matter for thought*

by A. K. Dewdney

"Nil posse creari de nilo."
—LUCRETIUS, *De rerum natura*

I was not surprised to receive, nearly a year ago, a long missive from someone who claimed to have invented a matter fabricator. After all, among those who write to me suggesting interesting ideas there are a few whose assertions do stretch credulity. But since the essence of science is an open mind (if not a fully ventilated one), I try not to dismiss such letters until I have read them to the end.

I am glad I did just that with this particular letter, because the inventor based his assertion on a legitimate mathematical result known as the Banach-Tarski paradox. Named for the Polish mathematicians who discovered it in the 1920's, the paradox reveals how under certain conditions an ideal solid can be cut into pieces and then reassembled into a new solid twice as large as the original one.

Indeed, the inventor turned out to be a professional mathematician who has many published papers to his credit. For reasons that will soon become clear, he shuns any kind of publicity and has asked that I call him Arlo Lipof. It was Lipof's familiarity with the Banach-Tarski paradox that first led him to investigate the possibility of applying the paradox to real matter instead of ideal matter. His investigation has paid off handsomely: he has written a computer program that gives precise directions for cutting a solid into many odd-shaped pieces and then reassembling them into a solid twice the size, leaving absolutely no space between the pieces!

Needless to say, the implications of Lipof's program are profound. To explain the paradox and how the program exploits it, I can hardly do better than to quote from Lipof's letter:

"The paradox is similar to the well-known puzzle involving tangrams, little pieces of paper cut into simple geometric shapes. Four such shapes can be assembled into a square that has an area of 64 square inches. Yet the very same pieces can also be assembled into a rectangle that has a greater area—65 square inches, to be precise. If you do not believe such a thing is possible, try cutting up the pieces as shown in my drawing" [*see illustration below*].

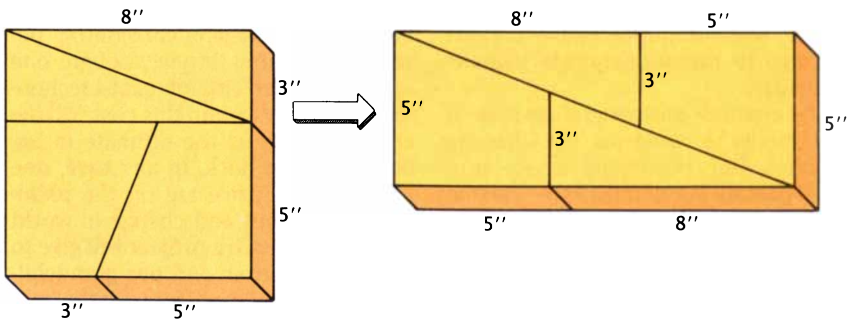
"If the little pieces of paper were instead pieces of gold, there would seem to be an automatic increase in wealth in going from the square to the rectangular configuration. Start with a square of gold, say, eight inches on a side and an inch thick. Then cut it according to the figure at the left. If the pieces are reassembled according to the figure at the right, an extra cubic inch of gold will have appeared. The extra cube would weigh approximately 4.3 ounces and at current prices would be worth about $1,800."

Lipof goes on to concede that the appearance of getting something for nothing in the above example is purely illusory. But he asserts that although the Banach-Tarski paradox has "the same effect on one's mind," there is no flaw in the theory on which it is based. The Banach-Tarski paradox is real—at least in a mathematical sense.

The paradox arises from a proved theorem that, when stated in technical language, is almost comprehensible: if $A$ and $B$ are any two bounded subsets of $\mathbf{R}^3$, each having a nonempty interior, then $A$ and $B$ are equidecomposable. The theorem can be stated in less technical language if one initially considers a pair of bodies, of virtually any shape and size, that meet two criteria. Each body must be "bounded," or capable of being enclosed in a hollow sphere of some definite size. And each must have a "nonempty interior": it must be possible to envision a sphere somewhere inside the body that is entirely filled with the material of which the body is made.

The two criteria are actually rather modest ones. Indeed, almost any object we might imagine that violates them is hardly the kind of object we would normally call a body. A straight, infinitely long line, for example, violates both criteria: it is not bounded and its interior is empty in the sense that it has no interior to speak of. Also disallowed would be an imaginary cloud of points stretching to infinity in all directions—hardly a body in the usual sense of the word.

According to the theorem, then, any two such bounded bodies having nonempty interiors are "equidecomposable." This means that one can dissect both bodies into a finite number of pieces that are congruent in a geometric sense: a piece of one body can be made identical with a piece of the other body merely by rotating it. Hence one can in theory dissect a body into pieces and label them $A_1, A_2, A_3 \ldots$ and dissect a different body into pieces and label them $B_1, B_2, B_3 \ldots$, so that the pieces $A_1$ and $B_1$, $A_2$ and $B_2$ and so on are identical. That is the essence of the Banach-Tarski paradox.




***How to get a cubic inch of gold for nothing***

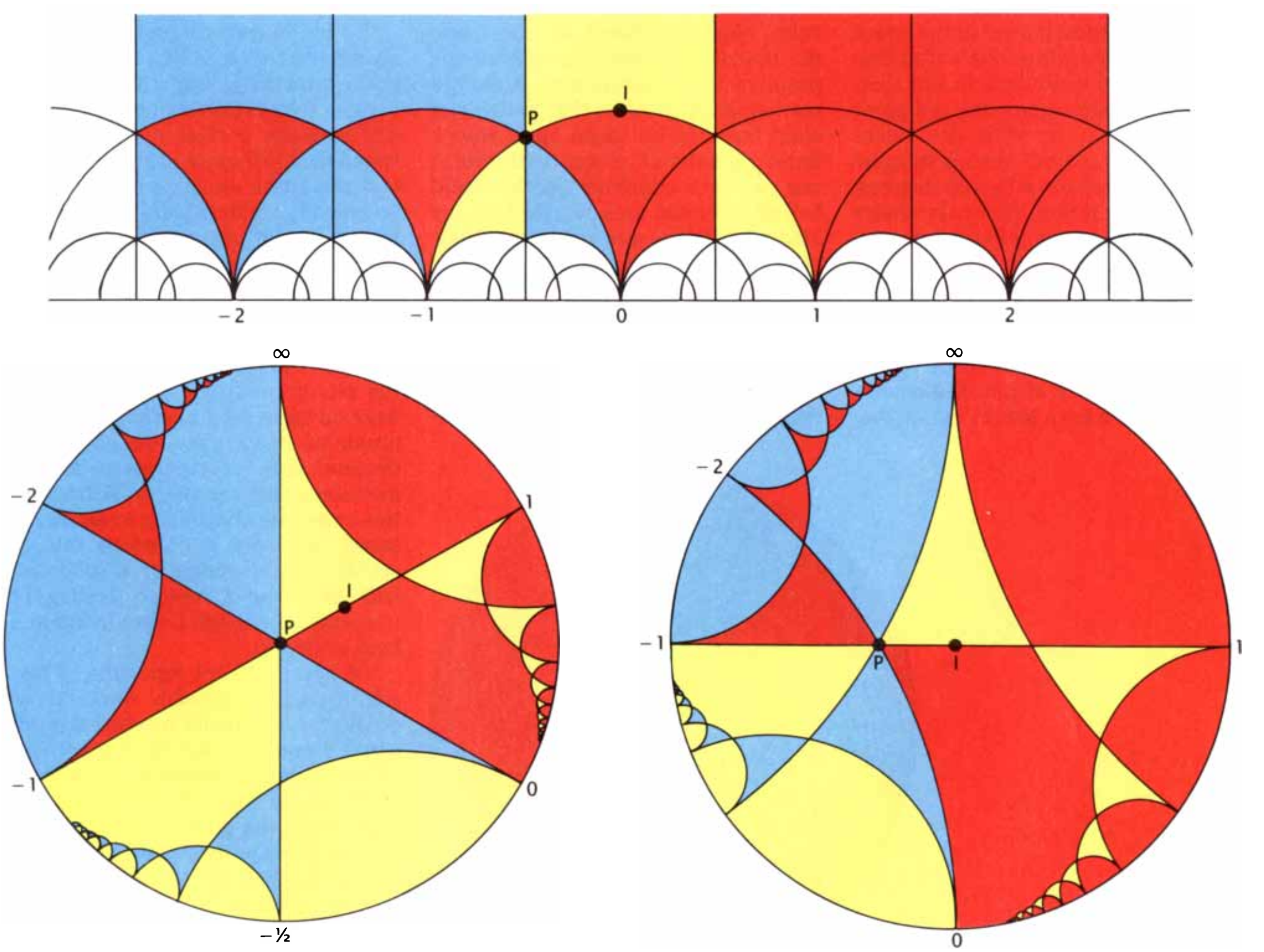
"It is thus possible," Lipof writes, "to take two solid spheres, one twice as large as the other, and cut them into pieces that are pairwise congruent. Forget the larger sphere and consider only the smaller one. Imagine that it is made of gold. In principle it can be cut into a finite number of pieces that can then be reassembled into a sphere twice as large."

There is no trickery here, although one must realize that there are certain topological qualifications implied in the innocent word "pieces." For one thing, they are not necessarily simple in form, or even composed of connected parts. Some parts of the same piece may come arbitrarily close to one another without actually touching. Moreover, the pieces cannot be measured in any precise way. For example, one cannot even imagine a way of gauging their exact volume. What would be the actual appearance of such pieces? Lipof says they are "like nothing you have ever seen before. They make fractals look like tangrams."

The Banach-Tarski paradox holds in its most general form only in spaces of three or more dimensions. There are, however, closely related theorems that illustrate the nature of the paradox in spaces of lower dimensions. A crude example of this phenomenon is given by the one-dimensional "space" consisting of all integers, since the subset of even integers represents simultaneously both half of and the whole of the set. The subset is half of the set of all integers in the sense that only every other integer is in the subset. Yet a simple transformation—dividing each element in the subset by 2—turns the even integers into the entire set of all integers; the set and its subset are the same size.

Most people do not find this fact very remarkable, because the size of both set and subset happens to be infinite. After all, infinity divided by 2 is still infinity. It would be more exciting to find a finite space that can be decomposed into paradoxical pieces, but that is not possible according to theory, at least if one limits oneself to spaces of a single dimension. The same applies for Euclidean spaces of two dimensions, or "flat" planes. It can happen, however, in certain non-Euclidean two-dimensional spaces.

A full explanation of the phenomenon is beyond the scope of this column, but I can give at least a glimpse of its paradoxical nature by projecting the exotic world of two-dimensional hyperbolic space onto an ordinary Euclidean disk, as is shown in the illustration below. The hyperbolic space occupies a half plane, as is shown in the upper part of the illustration. Its geometry is not Euclidean, wherein the shortest distance between two points is a straight line. Instead shortest distances are found along semicircles. In this illustration the hyperbolic space has been dissected into "triangular" regions that get smaller toward the bottom edge of the space. The triangles form the basis of a paradoxical decomposition of the space into three pieces that are colored red, blue




***Two projections of hyperbolic space* (above) *on disks* (below). *The red piece is both one-half and one-third of the hyperbolic space***

and yellow. To mathematicians, in this context a "piece" may not be all of a piece, so to speak. It may be composed of an infinite number of fragments, triangular and otherwise.

The strange nature of the three pieces becomes most apparent when one views the hyperbolic space through a special mathematical porthole, that is, by projecting the space in two different ways onto a disk. The point labeled *P* in the hyperbolic space lies at the center of the left-hand disk and the point labeled *I* lies at the center of the right-hand disk.

In each disk a simple rotational symmetry involving the three pieces becomes evident. Consider the red piece in the disk at the left. If one imagines rotating the disk about its center by 120 degrees, or a third of a revolution, one sees that it would end up—fragment for fragment—on top of the yellow piece. Another 120-degree rotation would similarly match the red piece with the blue one. In other words, all three pieces are congruent, and together the three pieces make up the entire hyperbolic space.

The paradoxical nature of the space becomes evident when one's attention turns to the second disk. In this view of the very same space the red piece is congruent to the other two pieces combined! To see this, merely imagine rotating the red piece by 180 degrees, or half a revolution, about the disk's center. The red piece will overlie exactly both the blue and the yellow piece. There is therefore a piece of two-dimensional hyperbolic space (the red piece) that amounts simultaneously to a half and a third of the entire space.

The significance of this demonstration may have been lost in the mathematical shuffle. "Why," asks the reader, "should I be impressed?" The reason is that the three sets represented by the colored pieces are absolutely true congruences in hyperbolic space. The fact that the red piece does not appear congruent to a half and a third of hyperbolic space at the same time on a single disk is a consequence of the distortions of hyperbolic space associated with such projections.

There is no need for hyperbole or hyperbolic spaces in proving the most general version of the Banach-Tarski paradox, however. The fact is that in Euclidean three-dimensional space (which approximates the world we live in) any two bodies that satisfy the most modest conditions imaginable are equidecomposable. Unfortunately the proof is nonconstructive: it gives almost no clue to precisely how one would go about demonstrating the equidecomposability of two unequal solid balls.

At this point I quote again from Lipof's letter:

"I spent many years studying the Banach-Tarski paradox and related results. What fascinated me most was the nonconstructive character of the proof in three dimensions. Although mathematicians know that in theory a solid ball can be taken apart into a finite number of pieces with which one can then construct another solid ball of twice the size, no one had any idea what the pieces might look like, because the cutting of the pieces is based on what set theorists call the axiom of choice.

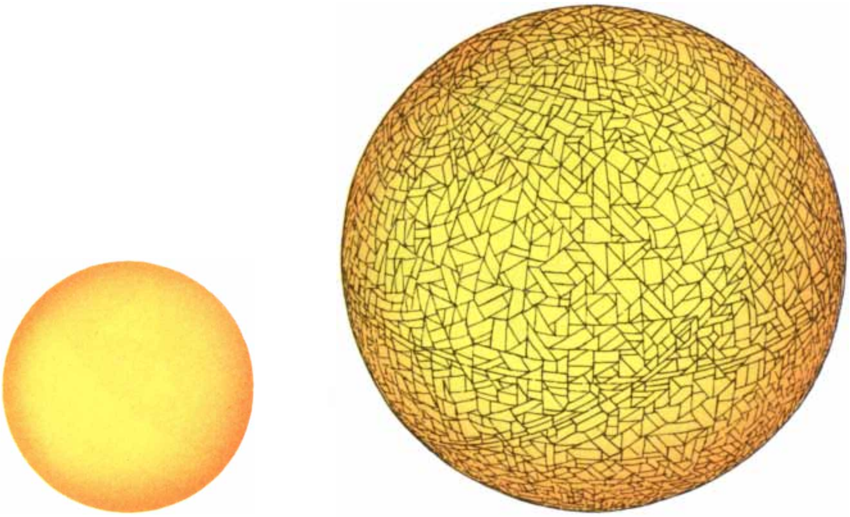
"The axiom gets its name not because mathematicians prefer it to other axioms but because it postulates that for any collection of sets, no matter how big, there is a way of choosing an element from each set in the collection. Indeed, many mathematicians prefer not to invoke the axiom, because it does not stipulate just how the matching is done.

"No one, therefore, had any idea of what the pieces of a paradoxical decomposition might look like until I began to investigate the question. I mechanized the proof of the Banach-Tarski paradox. The proof specifies that the second (larger) ball can be assembled by rotating the pieces of the first ball in two ways about its center, somewhat like the situation in hyperbolic space.... These rotations carry each piece of the first ball to its corresponding place in the second ball. Knowing the points that make up each piece and the necessary rotations, it was easy to construct a backtracking routine to cut the different pieces from a solid ball. Whenever the proof invoked the axiom of choice, I merely relied on a random-number generator in my personal computer to choose which points of the ball were to be elements in which sets.

"To be honest, throughout this research I had no idea that I was headed in the direction of a matter fabricator. I am no fool: I know that normally one has to make a clear distinction between the ideal spaces of mathematics and the space we live in. But when I completed my first simulation of the Banach-Tarski theorem, I realized I had in hand something like a recipe for doubling the size of any solid.

"It occurred to me to try an experiment involving a real material, but I initially held back. The dimensions of the pieces produced by my program were all expressed in triple precision numbers, an accuracy that might well demand that I cut atoms in two in manufacturing the pieces! Besides, at this stage I was beginning to doubt my sanity: the idea of carrying out an actual decomposition of a solid ball had given me a distinct feeling of unreality, as though I were living in a kind of dream.

"Of course, I kept repeating to myself, it couldn't possibly work. But to no avail. I eventually reached a point where I couldn't put the experiment off any longer. I cashed in a good part of my life savings to buy 12 ounces of gold. I had the gold cast into a ball, bought a tiny jeweler's saw and began to cut the ball up according to my program's recipe. A second computer program was most useful in this process. It catalogued the size and shape of each piece. In particular, the second program told me where each



***Original gold ball* (left) *and Arlo Lipof's reconstruction of it* (right)**

piece was to go in the second sphere.

"The entire experiment took seven months from start to finish. I worked nights and weekends. When at last I had finished cutting the pieces, I began assembling them into a second sphere of twice the diameter. It was delicate and demanding work. I nearly lost my eyesight and began to get headaches, but I persevered. Slowly the second ball took shape—but not as a smooth ball. The pieces did not fit as well as I had hoped. There were tiny spaces between the fragments I had so painfully and carefully put in place with tweezers.

"After a few more weeks I finally completed the ball. I have sent you a drawing of the major joints on its surface [*see illustration on opposite page*]. Having your readers see the map gives nothing away: the surface of the ball is child's play compared with the intricate arrangement of its interior pieces. In any event, the actual ball was not as smooth and round as my picture implies. It was lumpy and irregular—downright ugly. But how I clutched the cloth bag that contained it on my way to the jeweler! The ultimate test of all my work, of course, would be to melt the ball down and find out whether I was indeed the owner of up to eight times as much pure elemental gold as I had started out with.

"The next day the jeweler handed me a bar of pure gold weighing 49.58 ounces. It was less than I had expected; those interstitial spaces had taken their toll. Yet the thing was no longer in doubt. The world's first practical application of the Banach-Tarski paradox had been made. For days I staggered around like a drunk, reeling from my discovery. At this point I am not sure what to do next."

After that first letter I received no further correspondence from Lipof for several months. Then one day last November the mailman brought me a short missive from him, postmarked in a South American country:

"You will no doubt be happy to learn that I have to some extent automated the procedure of producing large balls of gold from small ones. With the remainder of my life savings I have set up shop in the little town of ________. Here a few loyal employees assemble gold balls. There is a workroom lined with computers and with tables at which my people assemble the balls. The pieces are now not cut out but rather are cast directly and worked by my employees. There is always excess gold at the end of the process with which to begin anew. We produce approximately five pounds of gold a week from nothing. Is this not the philosopher's stone?

"The time will soon come to move on. I do not think I will write again; to communicate with you is dangerous. Excuse me, my friend, but one becomes paranoid in the presence of such potential. There is much that I need to do!"

I have not heard anything more from Lipof. But last December, out of curiosity, I began to track the price of gold from day to day. For nearly three months it has been in a slow but steady decline. Perhaps that is the ultimate proof for those who thought the Banach-Tarski paradox was merely a plaything of mathematicians.

I have, of course, been in touch with other mathematicians on the subject of the paradox. I owe a particular debt of gratitude to Bruno W. Augenstein of the Rand Corporation in Santa Monica, Calif. It was Augenstein who suggested that I use hyperbolic space as an example of the paradoxical properties of space.

Although he does not subscribe to Lipof's claims, Augenstein does concede that there may well be a relation between the Banach-Tarski paradox and the real world. One of Augenstein's papers, "Hadron Physics and Transfinite Set Theory," points out a relation between particle physics and paradoxical decompositions of objects in three-dimensional space. The paper suggests analogies that "give directly a large number of known physical results and suggest additional ones testable in principle. The quark color label and the phenomenon of quark confinement... have immediate explanations via analogies with the decomposition theorems." This much might interest the physicists, if not the alchemists, among the readers of this column.

There are a few self-professed "bit flippers" in the world, readers who keep close watch for the appearance of a project that promises number-crunching complexity equal to their talents. The two-part series on cryptology last fall (in the October and November issues) raised a cheer in this quarter; in particular, the description of the Data Encryption Standard (DES) in the second part provided ample grist for a bit flipper's mill. The DES is a scheme for encrypting computer messages that is used not only by commercial institutions but also quite possibly by various military installations around the world. It is long and complicated, but that is just the way Mike Rosing of Darien, Ill., likes it. Eschewing software that is too soft, Rosing writes his own programs in 68000 assembly code, a low-level computer language that lies at the hardware heart of the 68000 microprocessor chip.

There is nothing like writing and testing a program (at any level) for revealing bugs in the original specifications. The input for the P permutation table in the F module was mislabeled; instead of 48 bits it should be 32 bits. Decryption also produced problems for Rosing and the others. The 64 bits of the original key are not fed in reverse order, the 48-bit "subkeys" are. They are fed into the central block starting with key 16 and ending with key 1.

Charles Kluepfel of Bloomfield, N.J., wondered what parts of the DES were arbitrary. For example, must the E bit-selection table really have the form I gave in order for a successful data-encryption system to emerge? And what about the substitution tables? Wafting rumors hold that the designers of the DES deliberately included "side doors" in certain parts of the cryptosystem that make it somewhat easier to decrypt DES ciphertext without knowing the original key.

Daniel Wolf of Santa Maria, Calif., has written several programs of interest in assembly code for his 68000-based Amiga computer. Amiga owners may request copies of a cryptosystem based on the famous Enigma machine (described in the October issue) or on the RSA algorithm (in the November issue). A major virtue of software cryptosystems that are written in assembly language is their blinding speed. Wolf can be reached at Box 1785, Santa Maria, Calif. 93456.

Still another journal of cryptology, *Cryptosystems Journal,* was brought to my attention by Tony Patti of Burke, Va., who edits and publishes it. The first two issues pursue Patti's goal of describing and distributing state-of-the-art cryptosystems for IBM PC's and compatible computers. Interested readers can get in touch with Patti for further information at 9755 Oatley Lane, Burke, Va. 22015.

### FURTHER READING

TANGRAMS—330 PUZZLES. Ronald C. Read. Dover Publications, Inc., 1965.

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THE BANACH-TARSKI PARADOX. Stan Wagon. Cambridge University Press, 1985.